A non-uniform node finite difference algorithm

***Abstract*—This thesis primarily investigates the application of the Finite Difference Method (FDM) in solving Ordinary Differential Equations (ODEs). Initially, the paper meticulously derives the FDM algorithm for uniform nodes, utilizing the perspective of Taylor expansion. Following this, the paper elucidates how the uniform node FDM algorithm can be employed to transform the problem of marginally-valued ordinary differential equations into a system of linear equations. Subsequently, the paper introduces non-uniform nodes into the FDM algorithm, reconstructs the finite difference, and proposes an FDM algorithm for non-uniform nodes. Finally, the paper employs MATLAB to simulate both the uniform node FDM algorithm and the non-uniform node FDM algorithm. The simulation results demonstrate that the accuracy of the non-uniform node FDM algorithm is ten times greater than that of the uniform node FDM algorithm, and it can fit the actual ODE results well.**

***Index Terms*—FDM, Boundary-valued ODE, Legendre polynomial, Numerical simulation,Numerical Analysis.**

I. Introduction

Numerical analysis is the science of algorithms for the study of continuous problems, mainly consisting of numerical methods for solving differential-integral and partial differential equations. The goal of numerical analysis is to design and analyse mathematical algorithms that in order to solve practical mathematical problems with limited accuracy and computational time [1].

The numerical solution of Ordinary Differential Equations (ODEs) is an important piece of numerical analysis, which mainly includes Euler's method, Improved Euler's method, and Lunger-Kutta's method.The core idea of these methods is to discretise the differential equations to create difference equations that give an approximation of the solution at some discrete points [2].

The solution of numerical ordinary differential equations is the basis of machine learning, and the effectiveness of machine learning is inseparable from the accuracy of numerical simulation, for example, the training process of neural networks can be regarded as an optimisation problem and the optimisation problem can be achieved by solving a set of numerical differential equations.

In practical engineering problems, solving ordinary differential equations with margin conditions is a common occurrence. However, ordinary differential equations often do not have stable analytical solutions, so we have to settle for the second best solution. The analytical solutions are obtained by numerical methods and applied. This thesis focuses on the FDM solution algorithm and proposes a non-uniform node FDM algorithm, which is verified by simulation to obtain higher numerical accuracy than the uniform node FDM algorithm.

II. Principles of FDM Algorithms

To make our discussion more concrete, we introduce the numerical integration problem that the thesis mainly addresses, Solving such differential equations:

$$u''(x) - b(n)u'(x) - c(n)u(x) = x^2 \tag{1}$$

and satisfies the conditions $u(-1) = -1, u(1) + u'(1) = 1$. where $x \in [-1, 1]$ and $b(n) = n^2, c(n) = n, n$ denotes the number of lattices in $[-1, 1]$, i.e. the lattice size is $h = \frac{2}{n}$. But it needs to satisfy $n \geqslant 32$.

This type of ordinary differential equations has the third type of edge-value condition [3], which is generalisable, and by studying its numerical solution, we can easily extend the numerical simulation method to the solution of other equations. According to the type division of ordinary differential equations, we can analyse this problem by looking at it as an initial value problem and a margin problem, and this paper focuses on the in-depth derivation and simulation of the solution of the margin problem.

The methods for solving the margin problem of ordinary differential equations mainly include the equivalent initial value problem solution method of the margin problem, the self-consistent solution method of the margin problem and the matrix form of the marginal value problem solution method [4], here mainly analyse the matrix form of the marginal value problem solution method.

Finite-difference methods (finite-difference methods, or FDM) [5], is a numerical method for differential equations that approximates derivatives by finite differences, the thereby seeking an approximate solution to the differential equation. The main idea is to approximate the derivative function by a linear combination of other conditions, and for simplicity and consistency of derivation. We use Taylor's formula for derivation. Two examples are given for the first-order difference, second-order difference, and the finite difference formula is derived by Taylor's expansion, and finally the first-order and second-order finite difference formula is given. Finally, we give common formulas for first-order and second-order finite differences and introduce finite difference matrices.

1) **First order finite difference**
   The main focus here is to derive the finite difference at the edges, and the derivation of the finite difference formula at the median is done in much the same way as the derivation of the finite difference at the edges [6].
   Suppose there is a set of functions $u(x)$ equidistant nodes $u_1, u_2, \cdots, u_n$, with the aim of finding the value of the derivative function at the point u_1 by finite difference, where the nodes are spaced Δx.

A Taylor expansion of the function $u(x_2)$ at the point u_1 is obtained:

$$u(x_2) = u(x_1) + \Delta x u'(x_1) + \frac{(\Delta x)^2}{2!} u''(x_1) + O((\Delta x)^3) \tag{2}$$

Take the first two terms of Taylor's formula and ignore all the subsequent terms, we could get a first-order finite difference formula with first-order accuracy:

$$u'(x_1) = \frac{u(x_2) - u(x_1)}{\Delta x} \tag{3}$$

To improve the finite difference accuracy, we can write the Taylor expansion of $u(x_3)$ at x_1 using values from more nodes:

$$u(x_3) = u(x_1) + 2\Delta x u'(x_1) + \frac{(2\Delta x)^2}{2!} u''(x_1) + O((2\Delta x)^3) \tag{4}$$

A linear combination $Au(x_2) + Bu(x_3)$ of Eq.(2) and Eq.(4) such that $\Delta x u'(x_1)$ has a coefficient of 1 (any one of these non-zero constants will do), and the coefficient of $(\Delta x)^2 u''(x_1)$ is 0, i.e.

$$\begin{cases} A + 2B = 1 \\ A\frac{1}{2!} + B\frac{2^2}{2!} = 0 \end{cases} \tag{5}$$

Finding $A = 2, B = -\frac{1}{2}$, then combining the two linearly according to their coefficients, and collapsing them, one gets

$$u'(x_1) \approx \frac{-3u(x_1) + 4u(x_2) - u(x_3)}{2\Delta x} \tag{6}$$

This is the first order finite difference formula with second order accuracy.

2) **Second-order finite difference** Here we derive the second-order difference at the median point $u(x_i)$, and similar to the procedure for the first-order difference, we respectively $u(x_{i+1})$ and $u(x_{i-1})$ perform a Taylor expansion at the point x_i:

$$\begin{aligned} u(x_{i+1}) &= u(x_i) + \Delta x'(x_i) + \frac{\Delta x^2}{2} u'(x_i) \\ &\quad + \frac{\Delta x^3}{3!} u''(x_i) + O\left(\Delta x^3\right) \\ u(x_{i-1}) &= u(x_i) - \Delta x'(x_i) + \frac{\Delta x^2}{2} u'(x_i) \\ &\quad - \frac{\Delta x^3}{3!} u''(x_i) + O\left(\Delta x^3\right) \end{aligned} \tag{7}$$

Similarly, a linear combination of them such that the coefficients in front of the second-order derivative be 1 and the coefficients in front of the third-order derivative be 0 gives the median form of the second-order finite difference:

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2} \tag{8}$$

Taking the above second-order median finite difference as an example, when dealing with the differential equation problem of discrete points by finite difference, the form of its components has an obvious regularity, and its second to $(n-1)$th rows are all only three values (their regularity is not covered by the fact that the first and last row elements of the differential matrix [7],which need to be determined by the edge-value condition) And the regularity is sorted in the order $\frac{1}{\Delta x^2}, \frac{-2}{\Delta x^2}, \frac{1}{\Delta x^2}$ as shown in the matrix below:

$$\frac{1}{\Delta x^2} \begin{bmatrix} \cdots & & \cdots & & \cdots \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \cdots & & \cdots & & \cdots \end{bmatrix} \tag{9}$$

Finally, the common formulas for first-order and second-order finite differences [8] are given as follows:

First-order derivative finite difference formulas with first-order accuracy:

$$\begin{cases} u'(x_0) \approx \frac{u(x_1) - u(x_0)}{h} \\ u'(x_1) \approx \frac{u(x_1) - u(x_0)}{h} \end{cases} \tag{10}$$

Finite difference formulas for first-order derivatives with second-order accuracy:

$$\begin{cases} u'(x_0) \approx \frac{1}{2h}\left[-3u(x_0) + 4u(x_1) - u(x_2)\right] \\ u'(x_1) \approx \frac{u(x_2) - u(x_0)}{2h} \\ u'(x_2) \approx \frac{1}{2h}\left[u(x_0) - 4u(x_1) + 3u(x_2)\right] \end{cases} \tag{11}$$

Finite difference formulas for the center value of second-order derivatives with second-order accuracy:

$$u''(x_1) \approx \frac{u(x_0) - 2u(x_1) + u(x_2)}{h^2} \tag{12}$$

The finite difference is thus briefly derived and exemplified, and the concepts of finite difference equations are introduced, which are next utilized to view the topic as a marginal value problem to be solved.

III. FDM matrix

Using finite difference matrices, Eq. (1) can be transformed into

$$D_2\vec{u} - n^2 D_1\vec{u} - nI\vec{u} = \overrightarrow{x^2} \tag{13}$$

$$\text{i.e. } (D_2 - n^2 D_1 - nI)\vec{u} = \overrightarrow{x^2} \tag{14}$$

where D_i denotes the finite difference matrix of order i, the linear combination of finite difference matrices is not directly equivalent to the original expression, due to the need to satisfy the margin conditions. Therefore, the differential matrix must be modified to satisfy the edge conditions before it can be used.

The edge conditions can be categorized into three types, but whichever edge condition is satisfied affects only the first and last rows of the finite difference matrix such that the linear equations satisfy the edge conditions of the ODE. Let $A = D_2 - n^2 D_1 - nI$. Because it is necessary to consider the first and last row elements of the linear combination of the differential matrix to satisfy the edge-value condition, the derivation ignores for the time being the first row elements and the last row elements to the process and place them all in ∞ as follows:

Second-order finite difference matrix

$$D_2 = \frac{1}{h^2}\begin{bmatrix} \infty & & \cdots & & \infty \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \infty & & \cdots & & \infty \end{bmatrix} \tag{15}$$

First-order finite difference matrix

$$D_1 = \frac{1}{2h}\begin{bmatrix} \infty & & \cdots & & \infty \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ \infty & & \cdots & & \infty \end{bmatrix} \tag{16}$$

Define the right end vector

$$\vec{b} = \vec{x}^2 = \begin{bmatrix} \infty \\ x_2^2 \\ \vdots \\ x_{n-1}^2 \\ \infty \end{bmatrix} \tag{17}$$

Next, the matrix is linearly combined so that the solution of the problem of ordinary differential equations is transformed into a system of linear equations, in order to make the solution process more explicit, the two ends of Eq.(14) are simultaneously Multiply by h^2,and we can obtain that:

Coefficient matrix h^2A

$$\begin{bmatrix} \infty & & & \cdots & & \infty \\ 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & & & \\ & \ddots & & \ddots & \ddots & \\ & & 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & \\ \infty & & & \cdots & & \infty \end{bmatrix} \tag{18}$$

The right end vector

$$h^2\vec{b} = \begin{bmatrix} \infty \\ h^2x_2^2 \\ \vdots \\ h^2x_{n-1}^2 \\ \infty \end{bmatrix} \tag{19}$$

Next the edge-value condition is judged such that the edge-value conditions $u(-1) = -1, u(1) + u'(1) = 1$ both hold.

The edge-value condition $u(-1) = -1$ is the Dirichlet edge-value condition, from which the row vector of the first row of h^2A can be determined as $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$,the first element of the corresponding right end vector is -1.

The edge condition $u(1) + u'(1) = 1$ is the Robin boundary condition [9], and a forward finite difference on $u'(1)$ has

$$\begin{aligned} &\frac{u_n - u_{n-1}}{h} + u_n = 1 \\ \Rightarrow &-u_{n-1} + (1+h)u_n = h \end{aligned} \tag{20}$$

The last row row vector of the coefficient matrix is thus determined as

$$[0, \cdots\cdots, 0, -1, h+1].$$

The last element of the corresponding right end vector is h.

After judging the margin conditions, we arrive at the coefficient matrix and the right end vector,see Eq. (21), and Eq. (22).

The right vector

$$h^2\vec{b} = \begin{bmatrix} -1 \\ h^2x_2^2 \\ \vdots \\ h^2x_{n-1}^2 \\ h \end{bmatrix} \tag{22}$$

Returning to the problem $Ax = (D_2 - n^2D_1 - nI)\vec{u} = \overrightarrow{x^2}$, which we dealt with at the beginning. This equation is transformed into $h^2Au = h^2b$, so that the ordinary differential equation for the marginals problem is transformed into a problem of solving a system of linear equations [10], and now we need only to complete the solution of this system of linear equations to obtain the numerical solution of the original differential equation at the equidistant node.

A. Optimizing the accuracy of uniform node FDM

In the above derivation process, two-point forward differencing is taken for the judgment of the edge-value condition $u(1)+u'(1) = 1$, which is of first-order precision, and we use second-order precision differencing in the second-order differential matrix and the second-order precision difference used in the first-order differential matrix does not match, which may lead to a final result of reduced precision [11], in order to optimize the problem, when dealing with the margin derivatives, we use a three-point forward differencing, i.e.

$$\left.\frac{du}{dx}\right|_{x=x_n} \approx \frac{3u(x_n) - 4u(x_{n-1}) + u(x_{n-2})}{2h} \tag{23}$$

The processing yields the row vector of the last row of the optimized coefficient matrix h^2A as

$$[0, \cdots\cdots, 0, 1, -4, 2h+3] \tag{24}$$

The last element of the corresponding right end vector is $2h$.

In this way the optimized coefficient matrix and the right end vector are obtained, as Eq. (25) and Eq. (26) shown.

$$\text{Coefficient matrix } h^2A = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & & \\ & \ddots & \ddots & \ddots & \\ & 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & \\ 0 & \cdots & 0 & -1 & h+1 \end{bmatrix} \tag{21}$$

$$\text{Optimization coefficient matrix } h^2A = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & & \\ & \ddots & \ddots & \ddots & \\ & 1+0.5nh^2 & -2-nh^2 & 1-0.5n^2h & \\ 0 & \cdots & 1 & -4 & 2h+3 \end{bmatrix} \tag{25}$$

The right vector

$$h^2\vec{b} = \begin{bmatrix} -1 \\ h^2x_2^2 \\ \vdots \\ h^2x_{n-1}^2 \\ 2h \end{bmatrix} \tag{26}$$

Using the optimized coefficient matrix and the right end vector, the value of the equidistant node with second order accuracy can be found.

IV. Non-uniform node FDM algorithm

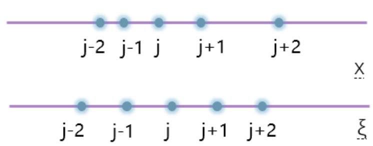

Fig. 1. One-dimensional grid

All of the above problems are differentiated for uniform grids, which may result in the Longueuil phenomenon [12], leading to a decrease in accuracy or even large fluctuations. Using non-uniform nodes for processing can effectively avoid the generation of the Longueuil phenomenon [13]and improve the accuracy of the ODE solution. Thus, non-uniform nodes can be sampled using non-uniform mesh and solved using non-uniform mesh finite difference.Fig.(1) shows a one-dimensional uniform network and a non-uniform grid, i.e., a numerical axis with equally spaced and unequally spaced points.

In this section, we still take the uniform way to prove finite difference, i.e., Taylor expansion, but the simplicity of the finite difference form for nonuniform nodes is far less than that for uniform nodes.In the following, the formula derivation of finite difference for non-uniform nodes is carried out and the focus is on constructing the first-order finite difference matrix and the second-order finite difference matrix for non-uniform nodes.

1) Non-uniform node first-order finite difference: The biggest difference between a non-uniform mesh and a uniform mesh is that the distance between each node may be different, so we cannot continue to use a uniform step size, but rather a specific problem for a specific analysis. Since the margin problem requires the construction of the first and last row vectors of the coefficient matrix, we only need to derive the finite difference at the intermediate node x_i. In the following, x_{i-1},x_{i+1} are Taylor-expanded at the node x_i, and the distance between x_{i-1} and x_i is h_1, and the distance between x_{i+1} and x_i is h_2.

$$\begin{aligned} u\left(x_{i-1}\right) &= u\left(x_i\right) - h_1u'\left(x_i\right) + \frac{h_1^2}{2!}u''\left(x_i\right) + o\left(h_1^3\right) \\ u\left(x_{i+1}\right) &= u\left(x_i\right) + h_2u'\left(x_i\right) + \frac{h_2^2}{2!}u''\left(x_i\right) + o\left(h_2^3\right) \end{aligned} \tag{27}$$

Do a linear combination of $Au(x_{i-1}) + Bu(x_{i+1})$ for both, so that the coefficients in front of the first-order derivatives are not zero, and the coefficients in front of the second-order derivatives are zero, to obtain the non-uniform node first-order finite difference formula, the solution can get the corresponding A and B, but here directly apply the original uniform node of the second-order accuracy of the first-order finite difference formula to get the non-uniform node of the first-order finite difference formula with second-order accuracy:

$$u'\left(x_i\right) = \frac{u\left(x_{i+1}\right) - u\left(x_{i-1}\right)}{h_1 + h_2} \tag{28}$$

The finite difference matrix can be obtained by constructing the finite difference formulas one by one by finding the distance between one node according to this formula. The subsequent non-uniform node solution is carried out in this way to construct the first-order finite difference matrix of non-uniform nodes.

We also need to derive the first-order finite difference for the right boundary condition due to the need of the edge value condition.

Assume that the boundary points of the rightmost segment are x_n,x_{n-1} and x_{n-2}, x_n, where the distance between x_{n-1} is h_n, and then the distance between x_{n-1},

x_{n-2} is h_{n-1}, and the distance between x_{n-2} is h_{n-2}. Taylor expansion at x_n for $u(x_{n-1})$ and $u(x_{n-2})$, respectively.

$$\begin{aligned} u\left(x_{n-1}\right) =& u\left(x_n\right) - h_n u'\left(x_n\right) + \frac{h_n^2}{2!} u^2\left(x_n\right) + o\left(h_n^2\right) \\ u\left(x_{n-2}\right) =& u\left(x_n\right) - \left(h_n + h_{n-1}\right) u'\left(x_n\right) \\ &+ \frac{\left(h_n + h_{n-1}\right)^2}{2!} u^2\left(x_n\right) + o\left(\left(h_n + h_{n-1}\right)^2\right) \end{aligned} \tag{29}$$

Make a linear combination of $C_1 u(x_{n-1}) + C_2 u(x_{n-2})$ for both such that the coefficients in front of the first-order derivatives are 1 and the coefficients in front of the second-order derivatives are 0,obtain:

$$\begin{cases} C_1 = -\frac{h_{n-1}+h_n}{h_n h_{n-1}} = -\frac{1}{h_n} - \frac{1}{h_{n-1}} \\ C_2 = \frac{h_n}{h_{n-1}(h_{n-1}+h_n)} \end{cases} \tag{30}$$

A first-order finite difference formula with second-order accuracy is thus obtained for the boundary points:

$$u'\left(x_n\right) = C_1 u\left(x_{n-1}\right) + C_2 u\left(x_{n-2}\right) - \left(C_1 + C_2\right) u\left(x_n\right) \tag{31}$$

The last row of the row vector of the coefficient matrix and the value of the last element of the right-hand side vector can be constructed for the edge-value condition by using Eq. (31), in exactly the same way as for the uniform node. The method is exactly the same as the construction of uniform nodes.

2) Non-uniform node second-order finite difference: Let the linear combination of $Au(x_{i-1}) + Bu(x_{i+1})$ for equation (27) be rearranged so that the coefficients of the first-order derivatives are 0 and the coefficients of the second-order derivatives are 1, yielding the linear combination coefficients:

$$\begin{cases} A = \frac{2}{h_1(h_1+h_2)} \\ B = \frac{2}{h_2(h_1+h_2)} \end{cases} \tag{32}$$

Then, we could obtain that

$$u''\left(x_i\right) = An\left(x_{i-1}\right) + Bn\left(x_1 + 1\right) - (A+B)u\left(x_i\right). \tag{33}$$

This finite difference formula has first-order accuracy, and the second-order finite difference matrix of non-uniform nodes can be obtained by using this finite difference formula to perform the construction one by one.

V. Numerical simulation

Two parallel routes are taken here, i.e., uniform node and non-uniform node finite difference to construct the numerical simulation for analysis.

A. Uniform node finite difference solution

In order to evaluate the accuracy of the solution obtained by the finite difference approach, we need to obtain a sufficiently accurate solution to the differential equation.The way to obtain an accurate numerical solution is to call the built-in function bvp4c in matlab for solving ordinary differential equations for margin problems.

The result of the program is shown in Fig. (2), and in order to visualize the difference between the exact solution

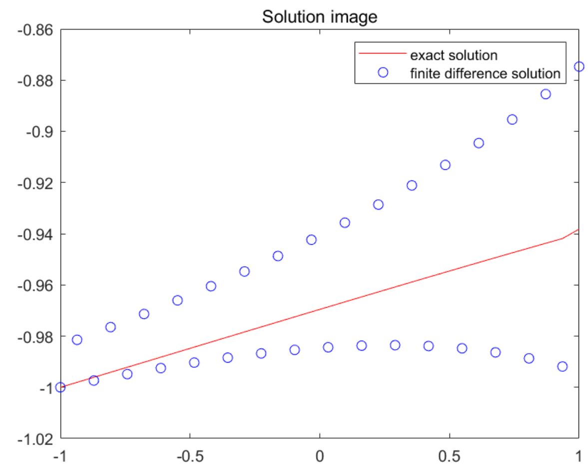

Fig. 2. Uniform Node FDM

and the numerical solution, we connect the exact solution with a smooth curve.

The variation curves of absolute and relative errors for uniform node finite difference are shown in Fig. (3), which shows that the numerical solution oscillates above and below the exact solution, the Moreover, the oscillation becomes gradually larger from the left endpoint to the right endpoint, and the exact solution can be approximated within the error range of 10%.

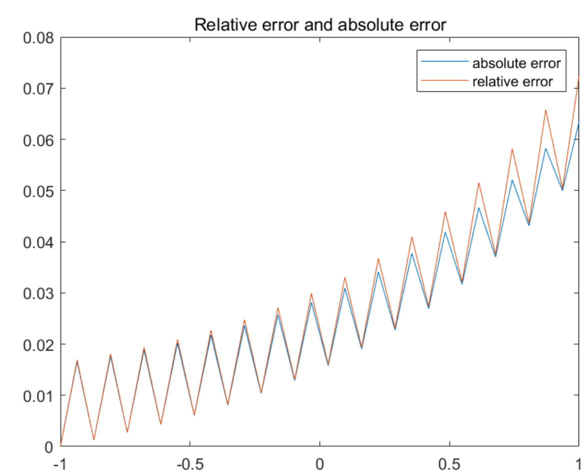

Fig. 3. Uniform Node FDM Error

B. Non-uniform node finite difference solution

Here, the nonuniform nodes use the 32 zeros of the 31-order Legendre polynomials, and in order to construct the edge-value condition, the first node is forced to be -1 and the last node is forced to be 1 to satisfy the mesh edge-value requirement.

The reason for choosing the zeros of Legendre polynomials as non-uniform nodes is because Legendre polynomials are orthogonal polynomials, which can effectively avoid the Lunge phenomenon and improve the accuracy and stability of numerical computation.

The result of the solution is shown in Fig.(4).

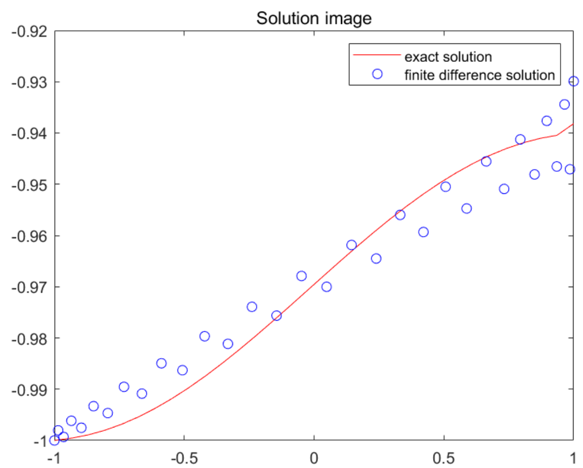

Fig. 4. Non-uniform Node FDM

The non-uniform node solution error is shown in Fig.(5).

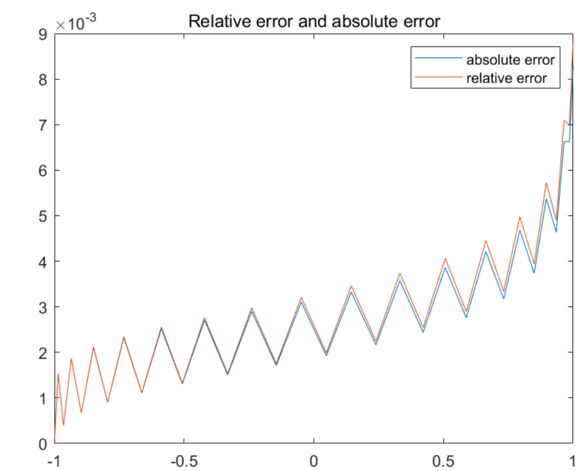

Fig. 5. Non-uniform Node FDM Error

According to the error image, it can be clearly seen that its relative error is at most no more than 1%, which is about ten times more accurate than the uniform node for solving this ordinary differential equation, and can fit the exact solution almost perfectly. After observing that the error also increases gradually from the left endpoint to the right breakpoint, which is consistent with the change of the uniform node, the reason is because of the instability of the boundary conditions at the right endpoint. This problem is not discussed in this article, so it will not be explained too much.

Comparing the finite difference accuracies of uniform and non-uniform nodes, it is easy to find that the accuracy of finite difference using orthogonal polynomial zeros as the non-uniform nodes is much greater than the accuracy of the difference of uniform nodes, which is verified in many algorithms of numerical analysis, the orthogonal bases play an important role in the representation of linear spaces and the improvement of numerical accuracy.

VI. Conclusion

Through our derivation and numerical simulation, using the zeros of Legendre polynomials as the nodes of the Non-uniform node FDM improves the accuracy of uniform node FDM by 10 times than uniform node FDM, and the relative error is in the range of around 1%, which can perfectly approximate the exact numerical solution of complex edge-value problems and is sufficient to meet the daily life's numerical accuracy demand, but because its computational complexity is higher than that of uniform nodes, the demand for computing power needs to be increased in practical applications, so that the computer can quickly process in real time.This enables the computer to quickly process the node numerical data in real time and complete the FDM operation of non-uniform nodes.

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