

复变函数

一. Complex Numbers and Complex Functions.

二. Analytic Functions.

三. Integral of Complex Function.

四. Complex Series.

五. Residues and Its Application

Analytic. 函数在 point 的某些邻域上解析, 该 point 解析.

Singular. 函数在某点不解析, 但该点所有邻域都有解析点.

isolated. 函数只在某点不解析, 孤立奇点.

一. Complex Numbers and Complex Functions.

$$\text{Arg } z = \begin{cases} \arctan \frac{y}{x}, & x > 0 \\ \arctan \frac{y}{x} + \pi, & x < 0, y \geq 0, \text{ 第二象限} \\ \arctan \frac{y}{x} - \pi, & x < 0, y < 0, \text{ 第三象限} \end{cases}$$

$$-\pi < \text{Arg } z \leq \pi$$

$$z \cdot \bar{z} = |z|^2$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\lim_{z \rightarrow z_0} f(z) = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) + i v(x,y) = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) + i \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = k$$

x, y 以任意方式逼近 (x_0, y_0) 均得到 k , 极限存在.

$$\lim_{z \rightarrow \infty} f(z) = \infty \Leftrightarrow \lim_{z \rightarrow \infty} \frac{1}{f(z)} = 0, \text{ 使用极坐标求极限较容易.}$$

二. Analytic Functions.

$$\left\{ \begin{aligned} f(z) = u(x,y) + i v(x,y) \text{ 有条件 (C-R equation)} \\ u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{aligned} \right.$$

$$\text{那么 } f'(z_0) = u'_x(x_0, y_0) + i v'_x(x_0, y_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

1. 数

$$\left\{ \begin{aligned} \text{利用极坐标变换, 若 } f(z) = u(r, \theta) + i v(r, \theta) \\ v_\theta = r u_r \\ u_\theta = -r v_r \end{aligned} \right.$$

$$\text{则有 } f'(z) = e^{-i\theta} (u_r(r, \theta) + i v_r(r, \theta))$$

~~邻域都解析~~
f 在一点解析, 只要在一点的某邻域解析.

孤立奇点, 只在一点处不解析.

邻域有解析点
f 在某一点不解析, 但周围点都解析, 则它称为奇点.

2. Harmonic functions.

$u_{xx} + v_{yy} = 0$, u 和 v 调和.

如果 $u(x,y)$ 和 $v(x,y)$ 都调和并满足 CR equation, v 是 u 的共轭 $\Rightarrow$ analytic.

3. Elementary functions.

$$\log z = \ln r + i(\theta + 2n\pi). \xrightarrow{\text{主值}} \text{Log } z = \ln |z| + i \text{Arg } z.$$

$\frac{d}{dz} \log z = \frac{1}{z}$, 但是有一个 cut branch 不连续.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

$$\text{反双曲} \quad z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow w = \sin^{-1} z.$$

三. Integral of Complex Function.

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt = W(t) \Big|_a^b.$$

1. 围线积分. $L = \int_a^b |z'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt. \quad (z = x(t) + iy(t)).$

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt. \quad \sim \text{计算非封闭路径.}$$

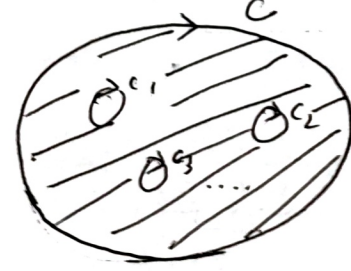
$$\int_C \frac{1}{(z-z_0)^{n+1}} dz = \begin{cases} 2\pi i, & n=0 \\ 0, & n \neq 0. \end{cases}$$

函数有原函数 $\Leftrightarrow$ 路径无关 $\Leftrightarrow$ 闭环积分为 0.

2. Cauchy integral theorem.

① 使用 C-R equation 判断 $f(z)$ 在闭环围线内解析, 则 $\int_C f(z) dz = 0$ (即域内解析).

② 解析域内路径无关.

③  $f(z)$ 在围线之中间隔处处解析, 有

$$\int_C f(z) dz = \int_{C_1} + \int_{C_2} + \int_{C_3} f(z) dz.$$

3. Cauchy integral formula.

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0). \quad \text{— 围线积分解只与奇点相关.}$$

4. Higher-order Cauchy integral formula

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

IV. Complex Series.

1. 绝对收敛 $\rightarrow$ 收敛.

2. Power Series.

收敛域判断. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| < 1 \Rightarrow |z|$ 在范围即收敛域.

$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n z^n|} < 1 \Rightarrow |z|$ 在范围即收敛域.

Abel's theorem. z_0 收敛, $|z| < |z_0|$ 绝对收敛.

z_0 收敛, $|z| > |z_0|$ 发散.

3. Taylor Series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!} (z-z_0) + \frac{f''(z_0)}{2!} (z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n + R_n(z).$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$$

$$(1+z)^{-1} = 1 + 2z + \frac{2(2-1)}{2!} z^2 + \dots$$

4. Laurent Series.

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n, \quad C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

求 Laurent Series, 一般使用间接法进行求解.

当 $n = -1$ 时, $C_{-1} = \frac{1}{2\pi i} \int_C \frac{f(z)}{1} dz$, 则 $f(z) = 2\pi i \underbrace{C_{-1}}_{Res}.$

五. Residues and Its Application.

1. singular point $\rightarrow$ isolated.

Three type of isolated singular points

对函数进行洛朗级数展开

如果在这点有有限级数, ~~essential singular point~~ pole of order k , $k=1 \Rightarrow$ simple pole.
 有无穷个负幂级数, essential singular point.
 无负幂级数, removed singular point.

将 pole of order m 的函数, $f(z) = \frac{p(z)}{(z-z_0)^m}$, $p(z_0) \neq 0$ 且 $p(z)$ 在 z_0 处 m 阶 pole.

但是, 函数列分母并不总是差值多项式.

引入零点, $f^{(m)}(z_0) \neq 0$ 但是 $f(z_0) = \dots = f^{(m-1)}(z_0) = 0$, z_0 是 zero of order m .

将零点, $f(z) = (z-z_0)^m p(z)$, $p(z_0) \neq 0$, $p(z)$ 在 z_0 处, zero of order m .

$\frac{f(z)}{p(z)}$, $f(z_0) \neq 0$, $p(z)$ 在 z_0 处 zero of order m , $\frac{f(z)}{p(z)}$ pole of order m .

(上述 $f(z_0) \neq 0$, 如果 $f(z_0) = 0$, 那么 $f(z)$ 在 z_0 处也有零点, 将零点以 $(z-z_0)^n p(z) = f(z)$ 的形式全部提取出, 再与分母抵消, 得到新的函数.)

2. Residues.

$$\int_C f(z) dz = 2\pi i C_1 = 2\pi i \operatorname{Res}_{z=z_0} f(z), [C-1, \text{洛朗级数取 } -1 \text{ 次幂前的系数}]$$

① 求洛朗级数, -1 次幂前边的系数即为 Res.

② Cauchy's residue theorem.

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z).$$



③ 奇点太多太多, 一个一个求十分困难, 寻找一种形式奇点.

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} [p(z)], p(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

(如果反过来求 0 不是 $f(z)$ 的奇点, $\operatorname{Res}_{z=0} f(z) = 0$).

④ 洛朗级数并不好展开 $\left(\frac{1}{(z-z_0)^m}\right)$ 的洛朗级数.

如果 $f(z)$ 在 z_0 处有 m 阶 pole, $f(z) = \frac{p(z)}{(z-z_0)^m}$.

$$\text{那么, } \operatorname{Res}_{z=z_0} f(z) = \frac{p^{(m-1)}(z_0)}{(m-1)!} \quad \leftarrow \text{使用高斯导数公式推导.}$$

⑤ 上述过程的特殊情况, 单重极点, 求解.

$f(z) = \frac{p(z)}{h(z)}$, $p(z_0) \neq 0$, $h(z)$ 在 z_0 处 1 阶零点, 或者说, $f(z)$ 在 z_0 处 Simple pole.

$$\text{那么有 } \operatorname{Res}_{z=z_0} = \frac{p(z_0)}{h'(z_0)}$$

工程及子 - 数学物理方法.

- 一、Equation of mathematical Physics and Problems for Defining Solutions.
- 二、Classification and Simplification for Linear Second order PDEs.
- 三、Integral method on Characteristics.
- 四、The method of Separation of Variable on Finite Region.
- 五、Special Function.
- 六、Integral Transformations.

一、数学物理方程.

$$\text{PDE条件} \left\{ \begin{array}{l} \text{初值} \left\{ \begin{array}{l} u|_{t=t_0} = f(\varphi) \\ u_t|_{t=t_0} = g(\varphi) \end{array} \right. \\ \text{边值} \left\{ \begin{array}{l} u(x,t)|_{x=0} = u(x,t)|_{x=l} = 0 \\ \text{② } \frac{\partial u}{\partial n}|_{x,y,z} = u_0 \\ \text{③ 混合②, ④} \end{array} \right. \end{array} \right.$$

$$\text{三类PDE} \left\{ \begin{array}{l} \text{① wave equation. } \frac{\partial^2 u}{\partial t^2} = a^2 \nabla^2 u + F(x,t) \\ \text{② heat equation. } \frac{\partial u}{\partial t} = a^2 \nabla^2 u + F(x,t) \\ \text{③ Laplace equation. } \nabla^2 u = F(x) \end{array} \right.$$

二、PDEs 的化简.

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G.$$

$$\rightarrow A \left(\frac{\partial y}{\partial x} \right)^2 - B \frac{\partial y}{\partial x} + C = 0.$$

$$\rightarrow \frac{\partial y}{\partial x} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}.$$

$$\Rightarrow \left\{ \begin{array}{l} \Delta = 0, \frac{\partial y}{\partial x} = \frac{B}{2A}, \text{ 得到一个 } \frac{\partial y}{\partial x}, \text{ 解出 } f(x,y) = \text{const}, \text{ 令 } \xi = y \text{ 即可.} \\ \Delta > 0, \text{ 得到两个方程 (ODEs), 解出 } f(x,y) = \text{const}, g(x,y) = \text{const}, \left\{ \begin{array}{l} \xi = f(x,y) \\ \eta = g(x,y) \end{array} \right. \\ \Delta < 0, \text{ 得到共轭复根, } \phi(x,y) \pm i\psi(x,y) = \text{const}, \text{ 令 } \left\{ \begin{array}{l} \xi = \frac{\psi + i\phi}{2} = \rho(x,y) \\ \eta = \frac{\psi - i\phi}{2i} = \varphi(x,y) \end{array} \right. \\ \text{即令实部和虚部分别为 } \frac{\phi(x,y)}{\xi} \text{ 和 } \frac{\psi(x,y)}{\eta} \end{array} \right.$$

随后利用偏导性质, 将变量全换为 ξ 和 η , 即可化简,

三、D'Alembert formula.

a) 无界弦振动.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & -\infty < x < +\infty, t > 0. \\ u(x, 0) = \phi(x), & -\infty < x < +\infty \\ u_t(x, 0) = \psi(x), & -\infty < x < +\infty. \end{cases}$$

$$\Rightarrow \begin{cases} \xi = x - at \\ \eta = x + at. \end{cases} \Rightarrow u_{\xi\eta} = 0 \Rightarrow \underline{u(\xi, \eta) = f(\xi) + g(\eta)}.$$

从而得 $u(x, t) = f(x - at) + g(x + at)$ 用初值条件判断

$$u(x, t) = f(x - at) + g(x + at) = f(\xi) + g(\eta).$$

$$u(x, t) = \frac{1}{2} (\phi(x - at) + \phi(x + at)) + \frac{1}{2a} \int_{x-at}^{x+at} \psi(z) dz.$$

b) 有界 (半无界) 弦振动 + 第一类边界条件.

i)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 < x < +\infty, t > 0 \\ u(x, 0) = \phi(x), & 0 \leq x < +\infty \\ u_t(x, 0) = \psi(x), & 0 \leq x < +\infty. \\ u(0, t) = 0, & t \geq 0. \end{cases}$$

半无界弦 + 边界条件 (第一类).

如果 D'Alembert formula 仍然适用.

$$\begin{cases} u(0, t) = \frac{1}{2} (\phi(-at) + \phi(at)) + \frac{1}{2a} \int_{-at}^{at} \psi(z) dz = 0. \\ \Rightarrow \phi(z) \text{ 和 } \psi(z) \text{ 均为奇函数.} \end{cases}$$

因此需要对 $\phi(x)$ 和 $\psi(x)$ 进行延拓 (奇延拓), 然后使用 D'Alembert formula. 并且在下面的求解中, 对 x 进行讨论.

ii) 半无界弦振动 + 第二类边界条件.

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 < x < +\infty, t > 0 \\ u(x, 0) = \phi(x), & 0 \leq x < +\infty \\ u_t(x, 0) = \psi(x), & 0 \leq x < +\infty \\ u_x(0, t) = 0, & t \geq 0. \end{cases}$$

继续使用 D'Alembert formula.

$$\begin{cases} \frac{1}{2} (\phi'(at) + \phi'(-at)) + \frac{1}{2a} (\psi(at) - \psi(-at)) = 0. \\ \text{那么 } \phi(x) \text{ 和 } \psi(x) \text{ 应都为偶函数, 才能满足条件.} \end{cases}$$

对原条件进行偶延拓, 随后进行讨论求解.

四、分离变量法.

a) 一维齐次方程.

i) 有界区域-一维弦振动.

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, t > 0. \end{array} \right.$$

$$u(0, t) = u(l, t) = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = g(x), \quad 0 \leq x \leq l.$$

$$\text{令 } u(x, t) = X(x)T(t). \Rightarrow \left\{ \begin{array}{l} \frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda. \end{array} \right. \quad \boxed{\lambda > 0}$$

$$\text{随后可得到 } \left\{ \begin{array}{l} T'' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0. \end{array} \right. \quad \text{从边界条件入手, 对 } \lambda \text{ 进行特征根讨论.}$$

得到 λ 的取值和对应的 $X(x)$, 再代入 $T(t)$ 进行求解.

随后线性组合 ~~$X(x)$ 和 $T(t)$~~ $X(x)T(t)$ 构成方程的解集.

随后代入初值条件进行求解, 变成 Fourier 级数的展开问题. (两次 Fourier 展开).

$$\text{ex. } \sum_{n=1}^{\infty} b_n \underbrace{\frac{\sin n\pi x}{l}}_{\text{Fourier 级数}} = g(x), \quad b_n \frac{\sin n\pi x}{l} = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

半波周期.

ii) 有界区域-一维热传导.

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, t > 0. \end{array} \right.$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(l, t) = 0, \quad t > 0.$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq l. \quad \boxed{\lambda > 0}$$

$$\text{先分离变量, 令 } \frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2 T(t)} = -\lambda \Rightarrow \left\{ \begin{array}{l} X''(x) + \lambda X(x) = 0. \\ T'(t) + \lambda a^2 T(t) = 0. \end{array} \right.$$

同样, 从边界条件入手, 讨论 λ 取值, 获得 $X(x)$ 和 λ 的系列值.

同样线性组合, 有.

$$\sum_{n=0}^{\infty} C_n \cos \frac{n\pi x}{l} = f(x). \quad \left\{ \begin{array}{l} C_0 = \frac{1}{l} \int_0^l f(x) dx. \\ C_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx. \end{array} \right. \quad \text{有时直接用 } C_n \text{ 不好求.}$$

$$\text{得到 } C_n, \quad u(x, t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \cos \frac{n\pi x}{l}, \quad \text{一次 Fourier 展开.}$$

6) 二维方程.

i) 矩形域 Laplace 方程.

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b. \\ u_x(0, y) = 0, u_x(a, y) = 0, & 0 \leq y \leq b \\ u(x, 0) = f(x), u(x, b) = 0, & 0 \leq x \leq a. \end{cases}$$

$\lambda \geq 0$

分离变量, 令 $\frac{x''}{x} = -\frac{y''}{y} = \lambda$, 随后挑选看起来比较简单的边界条件进行代入.

中途在讨论时, 可能出现 $y'' - \frac{\lambda}{4} y = 0$, 当时注意对 $n=0$ 的讨论.

$$\text{途中的 } u(x, y) = a_0 y + b_0 + \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{a} y} + b_n e^{-\frac{n\pi}{a} y}) \cos \frac{n\pi}{a} x.$$

ii) 圆形域 Laplace 方程.

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \left\{ \begin{array}{l} x = \rho \cos \theta \\ y = \rho \sin \theta \end{array} \right. \quad \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad \rho < R, 0 \leq \theta \leq 2\pi. \\ u(\rho, 0) = u(\rho, 2\pi), \quad \rho \leq R. \\ u(R, \theta) = a \cos \theta, \quad 0 \leq \theta \leq 2\pi. \\ u(0, \theta) < +\infty, \quad 0 \leq \theta \leq 2\pi. \end{array} \right. \end{cases}$$

$$\text{分离变量令 } \left\{ \begin{array}{l} \frac{\rho^2 P'(\rho)}{P(\rho)} + \frac{\rho P'(\rho)}{P(\rho)} = -\frac{(\Theta)''(\theta)}{(\Theta)(\theta)} = \lambda. \end{array} \right.$$

$$\boxed{\lambda \geq 0}, n=0, 1, 2, \dots$$

ODE.

$$\rho^2 P'(\rho) + \rho P'(\rho) - \lambda^2 P(\rho) = 0$$

欧拉方程

这类 ODE 如欧拉 ODE, 设 $P(\rho) = \rho^m$ 代入求得 m , 然后讨论性质.

$$\uparrow \text{上述方程讨论 } n=0 \text{ 时, } \rho^2 P'(\rho) + \rho P'(\rho) = 0 \Rightarrow [\rho P'(\rho)]' = 0 \Rightarrow P_0(\rho) = C_0 \ln \rho + D_0.$$

$$n \geq 1 \text{ 时, } P_n(\rho) = C_n \rho^{-n} + D_n \rho^n$$

$$\text{查 } \rho=0, \text{ 时 } u(0, \theta) < +\infty, \text{ 则 } C_0=0, C_n=0, P(\rho) = P_0 + \sum_{n=1}^{\infty} D_n \rho^n = \sum_{n=1}^{\infty} C_n \rho^n.$$

c) Sturm-Liouville eigenvalue problem.

$$f(x) = \sum_{n=1}^{\infty} f_n(x) u_n(x).$$

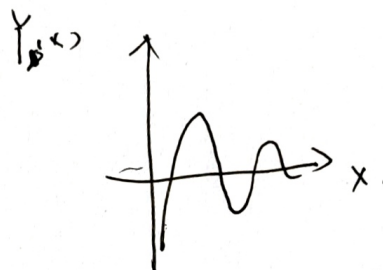
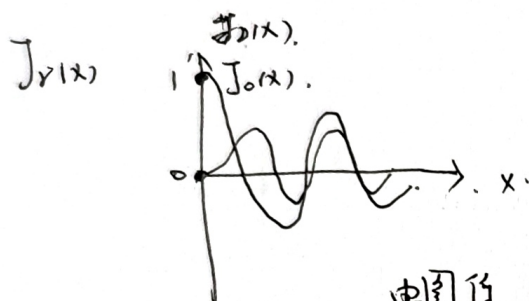
$$f_n = \frac{\int_a^b s(x) f(x) u_n(x) dx}{\int_a^b s(x) u_n^2(x) dx}, \quad (n=1, 2, \dots).$$

五. 特殊函数.

A. Bessel Function.

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \quad \nu \text{ 为 Bessel equation.}$$

$$\text{Solution: } y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$



由图可知, 只有 $J_0(x)|_{x=0} = 1$, 其它 $J_\nu(x)|_{x=0} = 0$.

$$J_\nu(x) \text{ 的性质 } \left\{ \begin{array}{l} \frac{d}{dx} [x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) \\ \frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x). \end{array} \right.$$

B. Legendre polynomial

$$(1-x^2) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + \nu(\nu+1)P(x) = 0.$$

如果 ν 不是整数. $\Rightarrow P(x) = a_1 P_\nu(x) + a_2 Q_\nu(x).$

如果 ν 是整数. $P_\nu(x) = C_1 \underline{P_n(x)} + C_2 \underline{Q_n(x)}.$
 第一类勒让德函数. 第二类勒让德函数.

$$P_n(x) \text{ 的性质 } \left\{ \begin{array}{l} P_0(x) = 1 \\ P_1(x) = x. \quad (2n+1)x P_n(x) \\ (n+1)P_{n+1}(x) - (2n+1)x P_n(x) + n P_{n-1}(x) = 0. \\ \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & n \neq m. \\ \frac{2}{2n+1}, & n = m. \end{cases} \end{array} \right.$$

六. 积分变换.

A. Fourier integral transformation.

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{i\lambda x} dx.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\lambda) e^{-i\lambda x} d\lambda.$$

B. Laplace integral transformation.

$$\left[\begin{aligned} L(s) &= \int_0^{+\infty} f(t) e^{-st} dt. \\ f(t) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} L(s) e^{st} ds. \end{aligned} \right] \quad \text{对时间进行 Laplace 变换.}$$

$$\left[\begin{aligned} f'(t) &\stackrel{L}{\longleftrightarrow} sL(s) - f(0^+). \\ f''(t) &\stackrel{L}{\longleftrightarrow} s^2 L(s) - sf(0^+) - f'(0^+). \end{aligned} \right]$$

$$t^n \stackrel{L}{\longleftrightarrow} \frac{n!}{s^{n+1}}$$

$$\cos kt \stackrel{L}{\longleftrightarrow} \frac{s}{s^2 + k^2}.$$

$$\sin kt \stackrel{L}{\longleftrightarrow} \frac{k}{s^2 + k^2}.$$

$$\int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$